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A METHOD FOR OBTAINING BENCHMARK NAVIER–STOKES SOLUTIONS

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SUMMARY

A refinement to an established method for obtaining benchmark Navier–Stokes solutions is presented. Pressure and body forces are derived explicitly such that the momentum equations are satisfied. The problem is reduced to determining a streamfunction in separation of variables form that describes a desired flow pattern. Examples based upon the well-known shear flow cavity are presented. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: incompressible flow; Navier–Stokes equations; benchmarking; exact solution

1. INTRODUCTION

Benchmarking, i.e. testing and validation of numerical flow codes, remains an important task in computational fluid dynamics [1–3]. An integrated approach, including comparison of independent results with corresponding numerical outputs, is employed by most practitioners. Analytical solutions play a role in this process, typically serving as a necessary but not sufficient condition for code approval. For example, because analyses are limited in terms of geometric and flow complexity, they usually act only as basic test cases and must be augmented by additional trials based upon other data, especially experimental measurements.

Each class [4,5] of analytical flow solutions has additional limitations with respect to the benchmarking problem. For example, a unidirectional flow configuration cannot be used for studying numerical convection schemes since the fluid motion is described by only a few viscous terms. Similarity solutions usually lack natural scales and therefore truncation of the numerical domain is necessary. Beltrami flows are periodic, again requiring an appropriately truncated domain. This class also does not admit realistic boundary conditions, such as the no-slip surface [6].

For the purpose of benchmarking, Shih [7] and co-workers [8] realized that adding a body force to the momentum equations as a degree of freedom allows for solutions on finite domains with no-slip boundary conditions. Several solutions based upon the shear cavity configuration introduced by Burggraf [9] were derived. In this note, an improved procedure for integrating the pressure is presented. Solutions for a cavity-like configuration are derived as examples.

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2. SOLUTION PROCEDURE

The stationary incompressible Navier–Stokes equations are cast in the following form:

$$\frac{\partial u_i}{\partial x_i}=0, \tag{1}$$

$$\frac{\partial u_i u_j}{\partial x_i}=-\frac{\partial P}{\partial x_j}+\frac{1}{Re}\cdot\frac{\partial^2 u_j}{\partial x_i\,\partial x_i}+h_j. \tag{2}$$

For two dimensions, Cartesian co-ordinates are $x_j=(x, y)$, velocity components are $u_j=(u, v)$, P is pressure, Re is the Reynolds number, and body force components are $h_j=(h_x, h_y)$.

As in Shih's method, a streamfunction in separation of variables form is assumed, $\psi=-f\cdot g$, where $f=f(x)$ and $g=g(y)$ are univariate 'kernel' functions. Velocity components are then given by $u=-f\cdot g'$ and $v=f'\cdot g$, where the prime symbol indicates differentiation. A desired flow pattern is obtained by choosing appropriate kernel functions. Furthermore, an arbitrary pressure distribution is specified in Shih's method. Body force h_j, the degree of freedom, must be determined by back-substituting velocity and pressure and solving the complete momentum equations, a tedious procedure. However, with the present method, both h_j and P are determined explicitly as functions of f and g, simplifying the problem. This occurs as follows.

In shorthand notation, the momentum equations can be expressed as $\partial P/\partial x=E+h_x$ and $\partial P/\partial y=F+h_y$, where $E=E(x, y, Re)$ and $F=F(x, y, Re)$ encompass convective and viscous effects. Integrating these expressions implies $P=\int(E+h_x)\,\mathrm{d}x=\int(F+h_y)\,\mathrm{d}y$. The body force must be chosen such that both expressions yield the same pressure distribution. This problem is underdetermined since there are two unknowns, the components of h_j, and only one equation. Therefore, h_j can be specified such that $\int E\,\mathrm{d}x=\int h_y\,\mathrm{d}y$ and $\int F\,\mathrm{d}y=\int h_x\,\mathrm{d}x$. These expressions can be shown to give the following results for the body force components and pressure distribution.

$$h_x=\frac{1}{Re}\cdot\left(f''\cdot g'+f''''\cdot\int g\,\mathrm{d}y\right)+\frac{(f\cdot f'''-f'\cdot f'')\cdot g^2}{2}, \tag{3}$$

$$h_y=-\frac{1}{Re}\cdot\left(f'\cdot g''+g''''\cdot\int f\,\mathrm{d}x\right)+\frac{(g\cdot g'''-g'\cdot g'')\cdot f^2}{2}, \tag{4}$$

$$P=\frac{(f\cdot f''-f'\cdot f')\cdot g^2+(g\cdot g''-g'\cdot g')\cdot f^2}{2}+\frac{1}{Re}\cdot\left(f'''\cdot\int g\,\mathrm{d}y-g'''\cdot\int f\,\mathrm{d}x\right). \tag{5}$$

3. EXAMPLE CONFIGURATION

A matching strategy is used to determine f and g for a given problem configuration. Boundary conditions are separated and the appropriate kernel function is evaluated, resulting in a set of constraint equations for f and g. As an example, a shear cavity problem is discussed.

The boundary is a square formed by $x=(0, 1)$ and $y=(0, 1)$, and the $y=1$ surface moves laterally. The four boundary surfaces and two velocity components yield a total of eight boundary conditions. Using the matching process, a total of eight constraint equations can be derived: $f=0$ and $f'=0$ for $x=(0, 1)$, $g=0$ for $y=(0, 1)$, $g'=0$ for $y=0$, and $g'\neq 0$ for $y=1$. Polynomials can be adapted to these constraints. For example, a third-order polynomial can be utilized for g because one of the constraints is non-trivial: $g=y^3-y^2$. For f, a fourth-order polynomial is used since all constraints are trivial: $f = -x^2 + 2x^3 - x^4$. All flow quantities can subsequently be derived.

$$\psi = (y^3 - y^2)\cdot(x^2 - 2x^3 + x^4), \tag{6}$$

$$\begin{aligned} P = &(-1.5y^4 + 2y^3 - y^2)\cdot(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4) \\ &+ (y^6 - 2y^5 + y^4)\cdot(-2x^6 + 6x^5 - 7x^4 + 4x^3 - x^2) \\ &+ \frac{(1-2x)\cdot(3y^4 - 4y^3) + 2x^3 - 3x^4 + 1.2x^5}{Re}, \end{aligned} \tag{7}$$

$$\begin{aligned} h_x = &(-2x + 12x^2 - 28x^3 + 30x^4 - 12x^5)\cdot(y^6 - 2y^5 + y^4) \\ &+ \frac{(3y^2 - 2y)\cdot(-2 + 12x - 12x^2) - 6y^4 + 8y^3}{Re}, \end{aligned} \tag{8}$$

$$h_y = (x^8 - 4x^7 + 6x^6 - 4x^5 + x^4)\cdot(-2y + 6y^2 - 6y^3) + \frac{(6y-2)\cdot(2x - 6x^2 + 4x^3)}{Re}. \tag{9}$$

The overall flow pattern is a single standing vortex, symmetric about $x = 0.5$. A quick check indicates that all boundary conditions are satisfied. Unlike the Burggraf configuration [9], there are no velocity singularities at the two junctions where the moving lid meets the fixed walls. The solution is entirely smooth. While many of the popular Navier–Stokes benchmarks are likewise smooth, caution must be used when applying these solutions to certain models, such as unstable finite elements, for which numerical problems may remain hidden. Moreover, accuracy and convergence for Equations (6)–(9) should not be taken as representative of Burggraf's physical problem.

Although ψ is independent of Re for this problem, higher-order expressions can be used to incorporate an explicit dependence upon the Reynolds number. For example, let f be described by the following fifth-order polynomial:

$$f = -x^2 + \frac{10x_0^3 - 12x_0^2 + 2}{d_f}\cdot x^3 + \frac{-5x_0^3 + 9x_0 - 4}{d_f}\cdot x^4 + \frac{4x_0^2 - 6x_0 + 2}{d_f}\cdot x^5. \tag{10}$$

The quantity d_f is $x_0\cdot(5x_0^2 - 8x_0 + 3)$ and $0 < x_0 < 1$ is an arbitrary location where $f = 0$. Equation (10) satisfies all of the constraints mentioned above.

Now assume that a solution valid for $0 < Re < 100$ is desired. The value $x_0 = 0.01\cdot Re$ can be substituted into Equation (10) to yield a new kernel function $f = f(x, Re)$. (Note that no flow is defined for $Re = 60$, since d_f vanishes at this value.) Different flow patterns develop depending upon Re. For example, at $Re = 50$, the motion forms a single vortex; however, at $Re = 80$, there are two weaker counter-rotating vortices.

4. CONCLUSION

A refinement to Shih's method of deriving solutions for code benchmarking has been presented. All flow quantities including pressure are given explicitly in terms of kernel functions. The problem is, therefore, reduced to deriving appropriate functions to obtain a desired flow pattern. A shear cavity configuration was used to show both Re-independent and Re-dependent examples.

While this method may show potential for practical usage, it is again emphasized that analytical solutions have important limitations with respect to the benchmarking problem.

Therefore, an integrated approach remains the best choice for reliably benchmarking numerical codes.

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